Journal of Engineering Mathematics, Vol. 5, No. 2, April 1971
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

An Approximate Nonlinear Dynamic Theory for Plates

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(Received March 17, 1970 and in revised form November 26, 1970)

SUMMARY
In this paper an approximate theory for the moderately large motion of transversely isotropic (orthotropic) plates is presented. Unlike other methods used in the past, the method used here involves no artificial assumptions. The method of approach begins with the equations of a partially non-linear elasticity theory and utilizes a method of asymptotic integration to arrive at successive two-dimensional approximations of increasing accuracy. The first approximation theory obtained here is the dynamic counterpart of Karman's plate theory. It includes the effect of rotatory inertia.

1. Introduction

This paper concerns itself with a systematic derivation of two dimensional approximations for the vibrations of transversely isotropic (orthotropic) plates from the known equations of a partially non-linear theory of three-dimensional elasticity. The method used is that of asymptotic integration which combines dimensional analysis with the expansion in powers of a small dimensionless parameter of the solution of the three-dimensional theory [1]. This technique was used in a previous article in this journal [2] to derive an iterative static large deflection theory for anisotropic plates.

For static problems the application of the method proceeds as follows: Plate dimensions are introduced via changes of the independent variables. The stresses and displacements are then non-dimensionalized and expanded in terms of a small geometric parameter. After introducing these expansions into the three-dimensional equations and equating equal powers of the parameter, sequences of systems of differential equations are obtained. The lowest order system represents the simplest approximate thin plate theory. The higher order systems incorporate thickness effects.

The extension of this method to dynamic problems is accomplished by introducing a length scale via the non-dimensionalization process and assuming that this length scale is also expandable in terms of the small geometric parameter. Depending on how one chooses this length scale, different two-dimensional stress states can be derived. The length scale is thus the characteristic wavelength associated with each theory. Expansion of the length scale allows for a correction of this characteristic wave-length due to the addition of higher order effects. For cylindrical shells this was previously demonstrated in [3].

Our attention in this paper is restricted to a length scale which is of the order of a characteristic length of the plate. Length scales of the order of the thickness h of the plate yield simplified elasticity equations. For static problems, these boundary layer stress-states need to be considered if a more exact approximation to the three-dimensional solution near the edges is desired (see, for example, [4]). For the problem of longitudinal wave propagation in cylindrical shells, it was shown in reference [3] that a theory corresponding to a length scale of $0(h)$ is needed to be able to obtain by superposition a wave velocity-wavelength spectrum valid for the whole range of wavelengths.

The first approximation equations obtained in this paper are the dynamic counterpart of the well-known von Karman equations [5]. Although the asymptotic method used here yields expressions for the transverse shear and normal stress components, their effect is absent in the stress-strain relations. The effect does appear in the equations of the second approximation. The effect of rotatory inertia, though, is present in the first approximation theory.

2. Basic Equations

Consider a plate made from a transversely isotropic (orthotropic) material and identified with a Cartesian coordinate system x_i $(i=1, 2, 3)$, such that $x_3=0$ represents the middle surface of the plate. Let the x_3-axis be the axis of symmetry. If the thickness of the plate is taken as $2h$, the plate then occupies a region bounded by two parallel surfaces $x_3=\pm h$ and a cylindrical surface having generators normal to the middle surface. It is assumed that the order of magnitude of the thickness is small compared to a representative length L along the cylindrical boundary. Within the framework of a partially nonlinear theory of elasticity [5], the governing equations can be written as

Strain-Displacement Relations

$$\varepsilon_{11} = u_{1,1}+\tfrac{1}{2}u_{3,1}^2, \qquad \varepsilon_{22} = u_{2,2}+\tfrac{1}{2}u_{3,2}^2, \qquad \varepsilon_{12} = u_{1,2}+u_{2,1}+u_{3,1}u_{3,2}$$
$$\varepsilon_{\alpha 3} = u_{\alpha,3}+u_{3,\alpha}, \qquad \varepsilon_{33} = u_{3,3} \tag{1}$$

where ε_{ij} denote the strain components and u_i the components of the displacement vector.

Equations of Motion

$$\sigma_{\alpha\beta,\alpha}+\sigma_{\beta 3,3} = \rho u_{\beta,tt}, \qquad \sigma_{\alpha 3,\alpha}+(\sigma_{\alpha\beta}u_{3,\beta})_{,\alpha}+\sigma_{33,3} = \rho u_{3,tt} \tag{2}$$

where σ_{ij} denote the stress components and ρ is the mass density.

Constitutive Equations

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11}-\nu\sigma_{22}) - \frac{\nu_3}{E_3}\sigma_{33}, \quad \varepsilon_{22} = \frac{1}{E}(\sigma_{22}-\nu\sigma_{11}) - \frac{\nu_3}{E_3}\sigma_{33}$$
$$\varepsilon_{33} = \frac{1}{E_3}\sigma_{33} - \frac{\nu_3}{E_3}(\sigma_{11}+\sigma_{22}), \quad \varepsilon_{12} = \frac{2(1+\nu)}{E}\sigma_{12}$$
$$\varepsilon_{13} = \frac{1}{G_3}\sigma_{13}, \quad \varepsilon_{23} = \frac{1}{G_3}\sigma_{23} \tag{3}$$

where E, E_3, G_3, ν and ν_3 are elastic constants. In the above, Latin indices range from one to three while Greek indices range from one to two. A comma preceding a subscript i denotes partial differentiation with respect to the coordinate.

For simplicity's sake the boundary conditions are taken as

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \qquad (x_3 = \pm h) \tag{4}$$

3. Asymptotic Approximation

We introduce dimensionless coordinates as follows:

$$\xi_\alpha = \frac{x_\alpha}{l}, \quad \xi_3 = \frac{x_3}{h}, \quad \tau = wt \tag{5}$$

where l is an as-yet unspecified length scale and w is the frequency. Dimensionless stresses and displacements are defined by

$$u_\alpha = \left(\frac{1-\nu^2}{E}\right)L\sigma v_\alpha, \qquad u_3 = \left(\frac{1-\nu^2}{E}\right)L\sigma\lambda^{-1}v_3 \tag{6}$$

$$\sigma_{\alpha\beta} = \sigma s_{\alpha\beta}, \quad \sigma_{\alpha 3} = \sigma \lambda s_{\alpha 3}, \quad \sigma_{33} = \sigma \lambda^2 s_{33}$$

where

$$\lambda = \frac{h}{L} \ll 1 \tag{7}$$

and σ is a representative stress level. In terms of these dimensionless variables, the fundamental equations can be rewritten as follows:

$$v_{3,3} = \frac{\lambda^2}{1-\nu^2}\left[\frac{\nu_n}{\nu_3}\lambda^2 s_{33} - \nu_n(s_{11}+s_{22})\right]$$

$$v_{1,3} = -\frac{1}{\mu}v_{3,1} + \left(\frac{2I_n}{1-\nu}\right)\lambda^2 s_{13}$$

$$v_{2,3} = -\frac{1}{\mu}v_{3,2} + \left(\frac{2I_n}{1-\nu}\right)\lambda^2 s_{23}$$

$$s_{11} - \nu s_{22} = \frac{1-\nu^2}{\mu}v_{1,1} + \nu_n\lambda^2 s_{33} - \frac{\gamma(1-\nu^2)^2}{2\mu^2\lambda^2}v_{3,1}^2$$

$$s_{22} - \nu s_{11} = \frac{1-\nu^2}{\mu}v_{2,2} + \nu_n\lambda^2 s_{33} - \frac{\gamma(1-\nu^2)^2}{2\mu^2\lambda^2}v_{3,2}^2$$

$$s_{12} = \left(\frac{1-\nu}{2}\right)\left[\frac{1}{\mu}(v_{1,2}+v_{2,1}) - \frac{\gamma(1-\nu^2)^2}{\mu^2\lambda^2}v_{3,2}v_{3,1}\right]$$

$$s_{13,3} = -\frac{1}{\mu}(s_{11,1}+s_{12,2}) + W^2 v_{1,\tau\tau}$$

$$s_{23,3} = -\frac{1}{\mu}(s_{12,1}+s_{22,2}) + W^2 v_{2,\tau\tau}$$

$$s_{33,3} = -\frac{1}{\mu}\left\{s_{13,1}+s_{23,2} + \frac{\gamma(1-\nu^2)}{\mu\lambda^2}\left[(s_{11}v_{3,1}+s_{12}v_{3,2})_{,1} + (s_{12}v_{3,1}+s_{22}v_{3,2})_{,2}\right]\right\} + \frac{W^2}{\lambda^2}v_{3,\tau\tau} \tag{8}$$

Here,

$$\mu = \frac{l}{L}, \quad W^2 = \left(\frac{1-\nu^2}{E}\right)w^2\rho L^2, \quad \gamma = \frac{\sigma}{E}$$
$$\nu_n = \frac{\nu_3}{E_3}E, \qquad I_n = \frac{E}{2(1+\nu)G_3} \tag{9}$$

The dimensionless variables are introduced in such a way that the order of all variables and their derivative is $0(1)$. We also consider only dimensionless groups of material constants which are $0(1)$. In view of condition (7), it is therefore reasonable to assume that we can expand the stresses and displacements in terms of a power series in λ^2 [see Eq. (8)],

$$s_{ij} = \sum_{m=1}^{M} s_{ij}^{(m-1)}\lambda^{2m}, \quad v_i = \sum_{m=1}^{M} v_i^{(m-1)}\lambda^{2m} \tag{10}$$

where $s_{ij}^{(m-1)}(\xi_k, \tau)$, $v_i^{(m-1)}(\xi_k, \tau)$ do not depend on λ for $m=1, 2, \ldots, M-1$ and the remainders

$s_{ij}^{(M)}(\xi_k, \tau; \lambda)$, $v_i^{(M)}(\xi_k, \tau; \lambda)$ are assumed to be such that they tend to a finite limit as λ approaches zero.

Length scale l is as-yet arbitrary. As it appears in Eq. (8) in terms of the ratio $1/\mu$, we assume that we can also expand this ratio in terms of a power series in λ^2,

$$\frac{1}{\mu} = \sum_{n=0}^{N} \mu^{(n)} \lambda^{2n} \tag{11}$$

where $\mu^{(n)} = 0(1)$. Form (11) implies $l = 0(L)$.

On substituting expansions (10) and (11) into Eq. (8) and equating corresponding powers of λ^2, successive systems of differential equations are obtained. The first two systems are:

First System

$$v_{3,3}^{(0)} = 0, \quad v_{1,3}^{(0)} = -\mu^{(0)} v_{3,1}^{(0)}, \quad v_{2,3}^{(0)} = -\mu^{(0)} v_{3,2}^{(0)}$$

$$s_{11}^{(0)} - \nu s_{22}^{(0)} = (1-\nu^2)\mu^{(0)} v_{1,1}^{(0)} + \frac{\gamma}{2}(1-\nu^2)^2 \mu^{(0)2} v_{3,1}^{(0)2}$$

$$s_{22}^{(0)} - \nu s_{11}^{(0)} = (1-\nu^2)\mu^{(0)} v_{2,2}^{(0)} + \frac{\gamma}{2}(1-\nu^2)^2 \mu^{(0)2} v_{3,2}^{(0)2}$$

$$s_{12}^{(0)} = \left(\frac{1-\nu}{2}\right)[\mu^{(0)}(v_{1,2}^{(0)} + v_{2,1}^{(0)}) + \gamma(1-\nu^2)^2 \mu^{(0)2} v_{3,1}^{(0)} v_{3,2}^{(0)}]$$

$$s_{13,3}^{(0)} = -\mu^{(0)}(s_{11,1}^{(0)} + s_{12,2}^{(0)}) + W^2 v_{1,\tau\tau}^{(0)}$$

$$s_{23,3}^{(0)} = -\mu^{(0)}(s_{12,1}^{(0)} + s_{22,2}^{(0)}) + W^2 v_{2,\tau\tau}^{(0)}$$

$$s_{33,3}^{(0)} = -\mu^{(0)}(s_{13,1}^{(0)} + s_{23,2}^{(0)} - \gamma(1-\nu^2)\mu^{(0)2}[(s_{11}^{(0)} v_{3,1}^{(0)} + s_{12}^{(0)} v_{3,2}^{(0)})_{,1} + (s_{12}^{(0)} v_{3,1}^{(0)} + s_{22}^{(0)} v_{3,2}^{(0)})_{,2}] + W_x^2 v_{3,\tau\tau}^{(0)}$$

Second System

$$v_{3,1}^{(1)} = -\left(\frac{\nu_n}{1-\nu^2}\right)(s_{11}^{(0)} + s_{22}^{(0)})$$

$$v_{1,3}^{(1)} = -(\mu^{(1)} v_{3,1}^{(0)} + \mu^{(0)} v_{3,1}^{(1)}) + \left(\frac{2I_n}{1-\nu}\right) s_{13}^{(0)}$$

$$v_{2,3}^{(1)} = -(\mu^{(1)} v_{3,2}^{(0)} + \mu^{(0)} v_{3,2}^{(1)}) + \left(\frac{2I_n}{1-\nu}\right) s_{23}^{(0)}$$

$$s_{11}^{(1)} - \nu s_{22}^{(1)} = (1-\nu^2)(\mu^{(1)} v_{1,1}^{(0)} + \mu^{(0)} v_{1,1}^{(1)}) + \nu_n s_{33}^{(0)} + \frac{\gamma}{2}(1-\nu^2)^2[\mu^{(1)2} v_{3,1}^{(0)2} + 4\mu^{(1)}\mu^{(0)} v_{3,1}^{(0)} v_{3,1}^{(1)} + \mu^{(0)2} v_{3,1}^{(1)2}]$$

$$s_{22}^{(1)} - \nu s_{11}^{(1)} = (1-\nu^2)(\mu^{(1)} v_{2,2}^{(0)} + \mu^{(0)} v_{2,2}^{(1)}) + \nu_n s_{33}^{(0)} + \frac{\gamma}{2}(1-\nu^2)^2 \times [\mu^{(1)2} v_{3,2}^{(0)2} + 4\mu^{(1)}\mu^{(0)} v_{3,2}^{(0)} v_{3,2}^{(1)} + \mu^{(0)2} v_{3,2}^{(1)2}]$$

$$s_{12}^{(1)} = \left(\frac{1-\nu}{2}\right)\{\mu^{(1)}(v_{1,2}^{(0)} + v_{2,1}^{(0)}) + \mu^{(0)}(v_{1,2}^{(1)} + v_{2,1}^{(1)}) + \gamma(1-\nu^2)^2 \times [\mu^{(1)2} v_{3,1}^{(0)} v_{3,1}^{(0)} + 2\mu^{(1)}\mu^{(0)}(v_{3,1}^{(0)} v_{3,2}^{(1)} + v_{3,1}^{(1)} v_{3,2}^{(0)}) + \mu^{(0)2} v_{3,1}^{(1)} v_{3,2}^{(1)}]\}$$

$$s^{(1)}_{13,3} = -\mu^{(1)}(s^{(0)}_{11,1}+s^{(0)}_{12,1})-\mu^{(0)}(s^{(1)}_{11,1}+s^{(1)}_{12,1})+W^2 v^{(1)}_{1,\tau\tau}$$

$$s^{(1)}_{23,3} = -\mu^{(1)}(s^{(0)}_{12,1}+s^{(0)}_{22,2})-\mu^{(0)}(s^{(1)}_{12,1}+s^{(1)}_{22,2})+W^2 v^{(1)}_{2,\tau\tau}$$

$$s^{(1)}_{33,3} = -\mu^{(1)}(s^{(0)}_{13,1}+s^{(0)}_{23,2})-\mu^{(0)}(s^{(1)}_{13,1}+s^{(1)}_{23,2})+W_x^2 v^{(1)}_{3,\tau\tau}-\gamma(1-\nu^2)$$
$$\times\{[\mu^{(1)2}(s^{(0)}_{11}v^{(0)}_{3,1}+s^{(0)}_{12}v^{(0)}_{3,2})+2\mu^{(1)}\mu^{(0)}(s^{(1)}_{11}v^{(0)}_{3,1}+s^{(0)}_{11}v^{(1)}_{3,1}+s^{(1)}_{12}v^{(0)}_{3,2}+s^{(0)}_{12}v^{(1)}_{3,2})+$$
$$+\mu^{(0)2}(s^{(1)}_{11}v^{(1)}_{3,1}+s^{(1)}_{12}v^{(1)}_{3,2})]_{,1}$$
$$+[\mu^{(1)2}(s^{(0)}_{12}v^{(0)}_{3,1}+s^{(0)}_{22}v^{(0)}_{3,2})+2\mu^{(1)}\mu^{(0)}(s^{(1)}_{12}v^{(0)}_{3,1}+s^{(0)}_{12}v^{(1)}_{3,1}+s^{(1)}_{22}v^{(0)}_{3,2}+s^{(0)}_{22}v^{(1)}_{3,2})$$
$$+\mu^{(0)2}(s^{(1)}_{12}v^{(1)}_{3,1}+s^{(1)}_{22}v^{(1)}_{3,2})]_{,2}\} \quad (13)$$

where

$$W_x^2 = \frac{W^2}{\lambda^2} \quad (14)$$

With regards to Eq. (12) and (13), the following is to be noted. Each system of equations can be integrated with respect to ξ_3 in a step by step manner. The asymptotic method used here leads to a solution giving all stress coefficients—inclucing those of transverse shear and normal stress. It is to be noted, however, that the transverse terms from the stress-strain relations do not appear in the first approximation system (12). Such terms first appear in the next system (13), as indicated by the presence of l_n and v_n. Scaling (14) indicates that the tangential frequency is of a higher order compared to the transverse frequency. Finally, it is necessary to expand length scale l due to the fact that the wavelength-frequency spectrum associated with a plate theory which includes second order effects must differ from that of the first approximation spectrum. The expansion of $1/\mu$ accomplishes this for each higher order systemyields a correction to the first approximation system (as shown in [3]).

In the following only the first approximation equations will be considered, as these yield the thin plate equations. The procedure for the higher order systems is analogous to that for the first, only lengthier.

4. First Approximation Theory

Integration of the first eight equations of (12) with respect to ξ_3 yields (omitting the superscripts on the stress and displacement coefficients):

$$v_3 = V_3(\xi_1, \xi_2, \tau) \quad (15)$$

$$v_1 = V_1(\xi_1, \xi_2, \tau)-\mu^{(0)}V_{3,1}\xi_3 \quad (16)$$

$$v_2 = V_2(\xi_1, \xi_2, \tau)-\mu^{(0)}V_{3,2}\xi_3 \quad (17)$$

$$s_{11} = \mu^{(0)}[V_{1,1}+\nu V_{2,2}+\frac{\gamma}{2}(1-\nu^2)\mu^{(0)}(V^2_{3,1}+\nu V^2_{3,2})-\mu^{(0)}(V_{3,11}+\nu V_{3,22})\xi_3] \quad (18)$$

$$s_{22} = \mu^{(0)}\left[\nu V_{1,1}+V_{2,2}+\frac{\gamma}{2}(1-\nu^2)\mu^{(0)}(\nu V^2_{3,1}+V^2_{3,2})-\mu^{(0)}(\nu V_{3,11}+V_{3,22})\xi_3\right] \quad (19)$$

$$s_{12} = \left(\frac{1-\nu}{2}\right)\mu^{(0)}[V_{1,2}+V_{2,1}+\gamma\mu^{(0)}(1-\nu^2)V_{3,1}V_{3,2}-2\mu^{(0)}V_{3,12}\xi_3] \quad (20)$$

$$s_{13} = S_{13}(\xi_1, \xi_2, \tau)-\mu^{(0)2}\left\{V_{1,11}+\left(\frac{1-\nu}{2}\right)\times V_{1,22}+\left(\frac{1+\nu}{2}\right)V_{2,12}-\frac{W^2}{\mu^{(0)2}}V_{1,\tau\tau}+\right.$$
$$\left.+\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)\left[(1-\nu)V_{3,1}\nabla^2V_3+\left(\frac{1+\nu}{2}\right)(V^2_{3,1}+V^2_{3,2})_{,1}\right]\right\}\xi_3$$
$$+\mu^{(0)3}\left[\left(\nabla^2V_3-\frac{W^2}{\mu^{(0)2}}V_{3,\tau\tau}\right)_{,1}\right]\frac{\xi_3^2}{2} \quad (21)$$

$$s_{23}=S_{23}(\xi_1,\xi_2,\tau)-\mu^{(0)2}\left\{V_{2,22}+\left(\frac{1-\nu}{2}\right)V_{2,11}+\left(\frac{1+\nu}{2}\right)V_{1,12}-\frac{W^2}{\mu^{(0)2}}V_{2,\tau\tau}+\right.$$

$$\left.+\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)\left[(1-\nu)V_{3,2}\nabla^2V_3+\left(\frac{1+\nu}{2}\right)(V_{3,2}^2+V_{3,1}^2)_{,2}\right]\right\}\xi_3$$

$$+\mu^{(0)3}\left[\left(\nabla^2 V_3-\frac{W^2}{\mu^{(0)2}}V_{3,\tau\tau}\right)_{,2}\right]\frac{\xi_3^2}{2} \tag{22}$$

where

$$\nabla^2(\ )=(\ )_{,11}+(\ )_{,22}$$

In terms of the dimensionless variables, boundary conditions (4) can be written as

$$s_{13}=s_{23}=s_{33}\qquad(\xi_3=\pm1) \tag{23}$$

Boundary conditions (23) are to be satisfied by each coefficient of expansions (10). Satisfaction of Eq. (23) by Eq. (21) and (22) yields.

$$S_{13}=-\frac{\mu^{(0)3}}{2}\left[\left(\nabla^2V_3-\frac{W^2}{\mu^{(0)2}}V_{3,\tau\tau}\right)_{,1}\right] \tag{24}$$

$$V_{1,11}+\left(\frac{1-\nu}{2}\right)V_{1,22}+\left(\frac{1+\nu}{2}\right)V_{2,12}-\frac{W^2}{\mu^{(0)2}}V_{1,\tau\tau} \tag{25}$$

$$=-\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)\left[(1-\nu)V_{3,1}\nabla^2V_3+\left(\frac{1+\nu}{2}\right)(V_{3,1}^2+V_{3,2}^2)_{,1}\right] \tag{25}$$

and

$$S_{23}=-\frac{\mu^{(0)3}}{2}\left[\left(\nabla^2V_3-\frac{W^2}{\mu^{(0)2}}V_{3,\tau\tau}\right)_{,2}\right] \tag{26}$$

$$V_{2,22}+\left(\frac{1-\nu}{2}\right)V_{2,11}+\left(\frac{1+\nu}{2}\right)V_{1,12}-\frac{W^2}{\mu^{(0)2}}V_{2,\tau\tau}$$

$$=-\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)\left[(1-\nu)V_{3,2}\nabla^2V_3+\left(\frac{1+\nu}{2}\right)(V_{3,1}^2+V_{3,2}^2)_{,2}\right] \tag{27}$$

Integration of the last equation of (12) now yields

$$s_{33}=S_{33}(\xi_1,\xi_2,\tau)+\frac{\mu^{(0)4}}{2}\left[\nabla^2\left(\nabla^2V_3-\frac{W^2}{\mu^{(0)2}}V_{3,\tau\tau}\right)\right]\left(\xi_3-\frac{\xi_3^3}{3}\right)$$

$$+W_x^2V_{3,\tau\tau}\xi_3-\gamma(1-\nu^2)\mu^{(0)3}\left\{V_{3,11}\left[V_{1,1}+\nu V_{2,2}+\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)(V_{3,1}^2+\nu V_{3,2}^2)\right]\right.$$

$$+V_{3,22}\left[V_{2,2}+\nu V_{1,1}+\mu^{(0)}\frac{\gamma}{2}(1-\nu^2)(V_{3,2}^2+\nu V_{3,1}^2)\right]$$

$$\left.+V_{3,12}(1-\nu)[V_{1,2}+V_{2,1}+\mu^{(0)}\gamma(1-\nu^2)V_{3,1}V_{3,2}]+\frac{W^2}{\mu^{(0)2}}(V_{3,1}V_{1,\tau\tau}+V_{3,2}V_{2,\tau\tau})\right\}\xi_3$$

$$+\frac{\gamma}{2}(1-\nu^2)\mu^{(0)4}[V_{3,1}\nabla^2V_{3,1}+V_{3,2}\nabla^2V_{3,2}+(\nabla^2V_3)^2$$

$$-2(1-\nu)(V_{3,11}V_{3,22}-V_{3,12}^2)]\frac{\xi_3^2}{2} \tag{28}$$

In order that boundary condition (23) be satisfied, we must have

$$S_{33} = -\frac{\gamma}{2}(1-\nu^2)\mu^{(0)4}[V_{3,1}\nabla^2 V_{3,1} + V_{3,2}\nabla^2 V_{3,2} + (\nabla^2 V_3)^2 - 2(1-\nu)(V_{3,11}V_{3,22} - V_{3,12}^2)] \quad (29)$$

$$\mu^{(0)4}\nabla^2\nabla^2 V_3 - W^2\mu^{(0)2}\nabla^2 V_{3,\tau\tau} + 3W_x^2 V_{3,\tau\tau}$$
$$= 3\gamma(1-\nu^2)\mu^{(0)3}\Big\{V_{3,11}\Big[V_{1,1} + \nu V_{2,2} + \mu^{(0)}\frac{\gamma}{2}(1-\nu^2)(V_{3,1}^2 + \nu V_{3,2}^2)\Big]$$
$$+ V_{3,22}\Big[V_{2,2} + \nu V_{1,1} + \mu^{(0)}\frac{\gamma}{2}(1-\nu^2)(V_{3,2}^2 + \nu V_{3,1}^2)\Big]$$
$$+ (1-\nu)V_{3,12}[V_{1,2} + V_{2,1} + \mu^{(0)}\gamma(1-\nu^2)V_{3,1}V_{3,2}] + \frac{W^2}{\mu^{(0)2}}(V_{3,1}V_{1,\tau\tau} + V_{3,2}V_{2,\tau\tau})\Big\} \quad (30)$$

Equations (25), (27) and (30) are the equations of motion for a partially non-linear plate theory which includes the effect of rotatory inertia. They represent the dynamic counterpart of von Karman's non-linear plate equations. The coefficient $\mu^{(0)}$ is to be determined from the relation

$$\frac{1}{\mu} = \mu^{(0)} \quad (31)$$

A higher order approximation can be determined from Eq. (13) by using the procedure used in the above analysis. This theory would incorporate the effects of transverse shear and normal stress, as indicated by the presence of I_n and v_n. In this case use has to be made of the relation (see [3])

$$\frac{1}{\mu} = \mu^{(0)} + \lambda^2\mu^{(1)} \quad (32)$$

If we set γ equal to zero in equations (25), (29) and (30), we obtain the equations of motion of the standard linear plate theory. The equations for the in-plane motion are now uncoupled from that for the transverse motion.

5. Stress Resultants

We define resultant forces and moments as follows:

$$N_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}\,dx_3 = \sigma h\int_{-1}^{1}s_{\alpha\beta}\,d\xi_3$$
$$M_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}x_3\,dx_3 = \sigma h^2\int_{-1}^{1}s_{\alpha\beta}\xi_3\,d\xi_3 \quad (33)$$

Dimensionless forces and moments are defined by

$$\bar{N}_{\alpha\beta} = \frac{N_{\alpha\beta}}{\sigma h} \qquad \bar{M}_{\alpha\beta} = \frac{M_{\alpha\beta}}{\sigma h^2} \quad (34)$$

If we now substitute the expansions for the stresses (10) and use Eq. (18), (19) and (20) the first approximation stress resultants are given by

$$\bar{N}_{11} = 2\mu^{(0)}\Big[V_{1,1} + \nu V_{2,2} + \frac{\gamma}{2}(1-\nu^2)\mu^{(0)}(V_{3,11}^2 + \nu V_{3,22}^2)\Big]$$

$$\bar{N}_{12} = (1-\nu)\,\mu^{(0)}[V_{1,2}+V_{2,1}+\gamma(1-\nu^2)\,\mu^{(0)}\,V_{3,1}\,V_{3,2}]$$

$$\bar{N}_{22} = 2\mu^{(0)}\left[\nu V_{1,1}+V_{2,2}+\frac{\gamma}{2}(1-\nu^2)\,\mu^{(0)}(\nu V_{3,11}^2+V_{3,22}^2)\right]$$

$$\bar{M}_{11} = -\tfrac{2}{3}\mu^{(0)2}(V_{3,11}+\nu V_{3,22})$$

$$\bar{M}_{12} = -\tfrac{2}{3}(1-\nu)\,\mu^{(0)2}\,V_{3,12}$$

$$\bar{M}_{22} = -\tfrac{2}{3}\mu^{(0)2}(\nu V_{3,11}+V_{3,22}) \tag{35}$$

The constant σ occurring in the above relations can be determined by the use of the boundary conditions. An example of this is given in [4].

6. Example

For small vibrations, equation (36) become

$$\mu^{(0)4}\nabla^2\nabla^2 V_3 - W^2\mu^{(0)2}\nabla^2 V_{3,\tau\tau}+3\,W_x^2\,V_{3,\tau\tau}=0$$

To determine the free vibrations of the plate, a solution of the form

$$V_3 = \bar{V}_3(\xi_1,\xi_2)\cos\tau \tag{37}$$

is assumed. On substitution into (36), one obtains

$$\mu^{(0)4}\nabla^2\nabla^2\bar{V}_3 + W^2\mu^{(0)2}\nabla^2\bar{V}_3 - 3\,W_x^2\,\bar{V}_3 = 0 \tag{38}$$

Consider now the special case of a clamped rectangular plate. Let l_1 be the length in the x_1 direction and l_2 the length in the x_2 direction. The edge conditions can then be stated as

$$\begin{aligned} \bar{V}_3 = \bar{V}_{3,1} = 0 \qquad & \left(\xi_1 = 0, \frac{l_1}{l}\right) \\ \bar{V}_3 = \bar{V}_{3,2} = 0 \qquad & \left(\xi_2 = 0, \frac{l_2}{l}\right) \end{aligned} \tag{39}$$

These conditions are satisfied by assuming $\bar{V}_3$ to have the form

$$\bar{V}_3 = A\,\sin^2\alpha_1\xi_1\,\sin^2\alpha_2\xi_2 \tag{40}$$

where

$$\alpha_1 = \frac{\pi l}{l_1}, \qquad \alpha_2 = \frac{\pi l}{l_2}$$

Use of the Galerkin method yields the following equation for the determination of the frequency:

$$\mu^{(0)4}(\alpha_1^4+\alpha_2^4+\tfrac{2}{3}\alpha_1^2\alpha_2^2) - \frac{W^2\mu^{(0)2}}{4}(\alpha_1^2+\alpha_2^2) - \tfrac{9}{16}W_x^2 = 0 \tag{41}$$

On substituting

$$\mu^{(0)} = \frac{1}{\mu} = \frac{L}{l}, \tag{42}$$

making use (9) and (14) and neglecting the effect of rotatory inertia, the following expression for the circular frequency w is obtained:

$$w^2 = \frac{8\pi^4 D}{3\rho h l_1^4}\left[1+\tfrac{2}{3}\left(\frac{l_1}{l_2}\right)^2+\left(\frac{l_1}{l_2}\right)^4\right] \tag{43}$$

where D is the flexural rigidity of the plate.

Acknowledgement

This work was supported in part by the National Science Foundation under Grant GK-3665 to the University of Illinois and by the Alexander von Humboldt Stiftung.

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Journal of Engineering Math., Vol. 5 (1971) 99–107